Bosonization theory of excitons in one-dimensional narrow-gap semiconductors

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(Received 15 November 2000; published 1 May 2001)

Excitons in one-dimensional narrow-gap semiconductors of anticrossing quantum Hall edge states are investigated using a bosonization method. The excitonic states are studied by mapping the problem into a nonintegrable sine Gordon type model. We also find that many-body interactions lead to a strong enhancement of the band gap. The system is modeled by the following Hamiltonian:

\[ H = H_0 + H_{\text{Coul}} + H_t, \]
\[ H_0 = \frac{\pi v}{\hbar} \int dx \left( -i \psi_R \partial_x \psi_R + i \psi_L \partial_x \psi_L \right) \]
\[ = \int dx \left( \rho_R^2 + \rho_L^2 \right), \]
\[ H_{\text{Coul}} = \int dx dy \left[ \frac{V(x-y)}{2} \left[ \rho_R^2(x) \rho_R^2(y) + \rho_L^2(x) \rho_L^2(y) \right] + V(x-y) \left[ \rho_R^2(x) \rho_L^2(y) \right] \right], \]
\[ H_t = -t \int dx \left[ \psi_R^\dagger(x) \psi_L^\dagger(x) + H.c. \right]. \]

The operator \( \psi_R \) (\( \psi_L \)) is the right-moving (left-moving) edge electron operator. The tunneling amplitude \( t \) is assumed to be significant only near the Fermi wave vector \( k_F \) (the Fermi wave vector is set to zero from now on). The cutoff value of the wave vector is \( 1/a \). Note that a single-particle gap opens up near the Fermi points due to this tunneling term. The commutation relations of density operators in momentum space are given by:

\[ [\rho_R(q), \rho_R(q')] = -\frac{q}{2\pi} \delta_{q+q'}, \]
\[ [\rho_L(q), \rho_L(q')] = +\frac{q}{2\pi} \delta_{q+q'}. \]
The interacting electron systems can be bosonized in a standard way. The explicit relations between the electron operators and the bosonic variables $\phi_R$ and $\phi_L$ are

\[ \rho_R = \frac{1}{\pi} \partial_x \phi_R, \quad \rho_L = \frac{1}{\pi} \partial_x \phi_L, \]

\[ \psi_R = \frac{1}{\sqrt{2\pi a}} e^{i\phi_R}, \quad \psi_L = \frac{1}{\sqrt{2\pi a}} e^{-i\phi_L}. \]

It is convenient to introduce the conjugate fields $\phi_\pm : \phi_\pm = \phi_R \pm \phi_L$. The effective bosonized action in imaginary time reads

\[ S = \int dx \, dv \left[ \frac{v}{8\pi} \left[ (\partial_x \phi_+)^2 + (\partial_x \phi_-)^2 \right] + \frac{i}{4\pi} \partial_t \phi_+ \partial_t \phi_- \right] \]

\[ + \frac{1}{8\pi^2} \int dx \, dy \, d\tau V(x-y) \partial_x \phi_+(x) \partial_x \phi_-(y) \]

\[ - \frac{i}{2\pi a} \int dx \, d\tau \left[ e^{i\phi_+(x)} + H.c. \right]. \quad (3) \]

The $\phi_-$ can be integrated out, leaving us with

\[ S = \int \frac{dk \, d\omega}{8\pi^2} \left[ \sqrt{k^2 \left( 1 + \frac{V(k)}{\pi v} \right) + \frac{\omega^2}{v}} \right] \phi_+(\omega, k) \]

\[ \times \phi_+(-\omega, -k) - \frac{i}{2\pi a} \int dx \, d\tau \cos[\phi_+(x, \tau)] \]. \quad (4) \]

The above action looks very similar to the sG model, except for the momentum-dependent Coulomb interaction $V(k)$. If $V(k)$ were momentum independent (local interaction in real space), the action would be exactly that of sG model. The Euclidean action of the sine Gordon model is given by

\[ A_{sG} = \int d^2x \left[ \frac{1}{16\pi} (\partial_\mu \phi)^2 - 2\mu \cos(\phi) \right]. \quad (5) \]

where the speed of “light” has been set to unity.

The explicit expression of the speed of light is $v = \sqrt{1 + V/\pi v}$, Note its dependence on $V$. It is useful to show the equivalence between the above sG action and our action (4) for the short-range interaction $V(k) \to V$ constant. This is achieved by the relations

\[ \phi = \phi_+ \left[ 4 \left( 1 + \frac{V}{\pi v} \right) \right]^{1/4}, \quad \beta = \left[ 4 \left( 1 + \frac{V}{\pi v} \right) \right]^{-1/4} \]

\[ \mu = \frac{t}{\pi av(1 + V/\pi v)^{1/2}}. \]

From the above relations it is clear that the strong-coupling regime of the original electron system (large $V$) is mapped to the weak-coupling regime (small $\beta$) of the sG model.

Before we investigate the long-range case it is instructive to review some known the physical properties of sG model, which is exactly solvable. The spectrum of sG model consists of the breathers ($B_n$), the soliton, and the antisoliton. The breathers are the bound states of the soliton and antisoliton. The lightest bound state is mapped to the soliton and the antisoliton. The lightest bound state is given by $n = 1, 2, 3, \ldots < 1/\xi$, where $\xi = \beta^2/(1 - \beta^2)$. Note that even for infinitesimally small short-range repulsion $V$, at least one breather exists. The breather mass (excitation energy) is given by

\[ m = 2M \sin \frac{\pi \xi}{2}, \quad (7) \]

where $M$ is the mass of soliton. The parameter $\mu$ and the soliton mass $M$ are related through

\[ \mu = \frac{\Gamma(\beta)}{\pi \Gamma(1 - \beta)^2} \left[ \frac{\sqrt{\pi \Gamma(\frac{1 + \xi}{2})}}{2} \right]^{2 - 2\beta^2}. \quad (8) \]

Note that in the $\beta \to 0$ limit, $M \sim v_p^2 \mu / \beta^2 \to \infty$ and $m \sim v_p^2 (\mu / \beta^2)^2 \beta^4 \sim u v / a$, where the speed of light $v_p$ has been reinstated for clarity. In other words, in the large-$V$ limit the soliton becomes very massive and leads to a large enhancement of the band gap. In contrast, the lightest breather mass approaches a constant value given by the coefficient of the cosine term.

In the absence of the Coulomb interaction the exact value of the single-particle gap should be $t$. However, the bosonized action (4) contains the factor $a$, and it is unclear how this factor disappears in the final result for the gap. Let us try to understand how this happens. The crucial fact is that the cosine term of the sG action (5) is normal ordered and it gives the dimension to the cosine operator. The exact results Eqs. (7) and (8), as obtained with

\[ \mu^2 : \cos[\beta \phi(x)] : \to \frac{1}{2} \frac{\mu^2}{|x-y|^2}, \quad (9) \]

as $|x-y| \to 0$. In the absence of Coulomb interaction, corresponding to $\beta^2 = 1/2$, the same correlation functions (two-point function of the tunneling term) can be computed exactly when expressed in terms of electron operators

\[ \langle [\psi_R(x) \psi_L(x)] [\psi_R(y) \psi_L(y)] \rangle = \frac{t^2}{|x-y|^2}. \quad (10) \]

The formal application of the bosonization formula to this electron tunneling term gives $(t/\pi a) \cos(\beta \phi)$ as in our action (4). As it stands it is not normal ordered, and consequently the short-distance singularity must be regularized in the calculation of correlation function:
We have to regularize \( \langle 0 | \phi(0) \phi(0) | 0 \rangle \) as \( \langle 0 | \phi(x) \phi(y) | 0 \rangle \) to get an effective action of exciton states: 

\[
\begin{align*}
\left( \frac{t}{2 \pi a} \right)^2 \langle e^{i \beta \phi(x)} e^{-i \beta \phi(y)} \rangle &= \left( \frac{t}{2 \pi a} \right)^2 \exp \left[ -\beta^2 \left( \langle \phi(0) \phi(0) \rangle - \langle \phi(x) \phi(y) \rangle \right) \right].
\end{align*}
\]

(11)

We have to regularize \( \langle \phi(0) \phi(0) \rangle \) as \( \langle \phi(0) \phi(a) \rangle \) for nonintegrable \( L \) where \( L \) is the system size. \( L \) always appears in the definition of propagator but it is canceled for the physical correlation functions (see below). Since \( \langle \phi(x) \phi(y) \rangle \) is the system size,

\[
\begin{align*}
\left( \frac{t}{2 \pi a} \right)^2 \langle e^{i \beta \phi(x)} e^{-i \beta \phi(y)} \rangle &= \left( \frac{t}{2 \pi a} \right)^2 \exp \left[ -\beta^2 \left( \langle \phi(t=0) \phi(0) \rangle - \langle \phi(x) \phi(y) \rangle \right) \right].
\end{align*}
\]

(12)

Only at \( \beta^2 = 1/2 \) does the length scale drop out. Comparing Eq. (9) with Eq. (12), \( t \) can be identified with \( \mu \). Then by applying Eq. (8) at \( \beta^2 = 1/2 \), we find that \( t = M \), which is the expected result. The above argument checks the consistency of the bosonization formulation of our problem. We emphasize again that the results above hold only if the Coulomb interaction is absent.

When the long-range Coulomb interaction is present the model becomes nonintegrable and an exact solution is unavailable. But, under certain conditions, the cosine term of the action (4) can be expanded and excitons may be studied perturbatively. (See the perturbative calculation below.) Expanding the cosine term of Eq. (4) up to the fourth order, we get an effective action of exciton states:

\[
S = \frac{1}{2} \int \frac{dk}{(2 \pi)^2} \left( \frac{k^2}{\pi \hbar^2} + \frac{\hbar^2}{\pi \hbar^2} + \frac{4t}{\hbar} \sum_{\gamma} \right)
\]

\[
N^2 \left( \phi^4 \right) - \int dx dt \phi^4.
\]

(13)

At this point, we note that the parameter \( t/\pi a \) in action (4) should be understood as a renormalized quantity. The application of bosonization formula to the fermion bilinear mass term assumes the implicit normal ordering.\(^\text{10}\) Such a normal ordering sums the tadpole diagrams\(^\text{11}\) (see Fig. 1) and the effect of the normal ordering is contained in the cutoff parameter \( 1/2 \pi a \). Therefore, in the perturbative expansion of cosine term we have to exclude all the tadpole type diagrams, and then all other terms of perturbative expansion are finite and well defined. In analogy with the sG model, we will associate the solitonlike (quantum) solution of Eq. (4) with the renormalized electrons and the breather like \( \phi^4 \) mode with the excitonic electron-hole bound state.

If the quantum correction due to \( \phi^4 \) is neglected, the zeroth-order dispersion relation of exciton state can be read off from the quadratic part of Eq. (13):

\[
E^{(0)}(k) = \hbar \left[ v^2 k^2 \left( 1 + \frac{v_0}{2v} \ln \frac{1}{a|k|} \right) + \frac{4tv}{\hbar a} \right]^{1/2}.
\]

(14)

The energy gap is given by

\[
\Delta_{\text{ex}} = \frac{\sqrt{4 \hbar v a}}{a} = 2 \sqrt{\alpha W}.
\]

(15)

Next, we compute the correction to the exciton excitation energy due to the quantum fluctuation of \( \phi^4 \) term. The first such correction appears in the second-order perturbation of \( \phi^4 \). Because the \( \phi^4 \) term is normal ordered there are no tadpole diagrams. Only the “sunset” diagram (see Fig. 1) contributes, which is both ultraviolet and infrared convergent. It contributes a negative correction to the gap as can be checked by a simple calculation \[ e^{-\Delta_{\text{ex}}^{(0)} v \phi^2} (1 + \lambda^2 \phi^2 (\phi^2 \phi^4) - e^{-t \phi^4}) \phi^4, \]

where \( \lambda = t/\pi \hbar a \).

The symmetry factor of the sunset diagram is \( (4!)^2/(4!)^4 \times 4 \times 3 \times 2 = 1/6 \). In terms of self-energy the sunset diagram gives

\[
\Sigma(i \omega, k) = \frac{1}{6} \left( \frac{t}{\pi \hbar a} \right)^2 \int dx d \tau [D(x, \tau)]^3 e^{i k x + i \omega \tau},
\]

\[
D(x, \tau) = \int \frac{d \omega}{(2 \pi)^2} \frac{v e^{i k x + i \omega \tau}}{\omega^2 + v^2 k^2 \left( 1 + (a/2 \hbar v) \ln(1/k) \right) + 4 \hbar v a}.
\]

(16)

The excitation energy, which includes the quantum correction, above is given by

\[
\Delta_{\text{ex}} = \left[ \frac{\hbar v a}{-8 \pi v^2 \Sigma(0, 0)} \right]^{1/2}.
\]

(17)

Write the above equation in the form

\[
\Delta_{\text{ex}}^2 = \left( \Delta_{\text{ex}}^{(0)} \right)^2 \left[ 1 - \frac{\Sigma(0, 0)}{t \left( 2 \pi \hbar a \right)} \right].
\]

(18)

The second term in the bracket of Eq. (18) is the quantum correction in the dimensionless form. Explicitly \( (\eta_1, \eta_2, \eta_1, \eta_2) \) are dimensionless.

\[
\delta_{\text{ex}} = \frac{\Sigma(0, 0)}{t \left( 2 \pi \hbar a \right)} = \int d \eta_1 d \eta_2 [D(\eta_1, \eta_2)]^3,
\]

\[
D(\eta_1, \eta_2) = \int \frac{d \eta_1 d \eta_2}{(2 \pi)^2} \frac{e^{i \eta_1 + i \eta_2}}{4 + \eta_1^2 + \eta_2^2}.
\]

(19)
where \( f(\eta_2) = 1 + (v/2\nu)\ln((1/\eta_2)) \times \sqrt{W/t} \). We may extract the effective expansion parameter by rescaling. It is easy to see that

\[
\delta_{\text{ex}} \sim \frac{\Sigma(0,0)}{t(2\pi\hbar a)} \sim \frac{v/\nu_0}{\ln W/t} \tag{20}
\]

We note that, in the perturbative regime \( \delta_{\text{ex}} \approx 0 \) the excitation energy approaches the value of the coefficient of the cosine term. This same feature also appeared in the exactly solvable sine Gordon model. Our perturbative treatment for the excitation states is thus valid when \( \delta_{\text{ex}} < 1 \). This result suggests the possibility that when \( \delta_{\text{ex}} > 1 \) is satisfied exciton instability may occur. However, we cannot exclude the possibility that the higher-order corrections neglected in our perturbative calculation may prohibit such an instability.

The action Eq. (4), supports topological soliton excitations due to the topologically inequivalent vacuum of the cosine potential. Thus, we cannot use the perturbative expansion of the cosine term in the study of the soliton excitation. The original sine Gordon model can be solved exactly, and the energy gaps of soliton and breather are known exactly [see Eqs. (7) and (8)]. Our action (4) is not exactly solvable, and here we will give only an estimate of the gap of soliton excitation. In estimating the gap of soliton excitation, it is sufficient to consider the static soliton. In our perturbative regime we may neglect \( 1 \) compared to \( V(k)/\pi \nu \). Then, in real space, the energy functional for static solution reads

\[
E_{\text{ex}}[\phi_+(x)] = \frac{1}{8\pi} \left[ \int dx dy \frac{\epsilon^2}{\epsilon|x-y|} \partial_x \phi_+(x) \partial_y \phi_+(y) \right]^{1/2} + \frac{t}{\pi a} \int dx \{ 1 - \cos[\phi_+(x)] \} \tag{21}
\]

(a constant term is added to make the expression positive definite). We define a dimensionless variable \( \tilde{x} = x(t/\epsilon^2 \pi a)^{1/2} = x(t/\epsilon a)^{1/2}(1/\pi) \). Then, the energy functional becomes

\[
E_{\text{ex}}[\phi_+(x)] = \sqrt{\frac{\epsilon^2 \pi}{\epsilon a}} \left[ \int dx dy \frac{1}{|x-y|} \partial_x \phi_+(x) \partial_y \phi_+(y) \right]^{1/2} + \frac{1}{\pi a} \int dx \{ 1 - \cos[\phi_+(x)] \} \right]. \tag{22}
\]

The coefficient in front of the large bracket has the dimension of energy, and it gives a characteristic gap scale of solitons. The ground state, which sits at a minimum of the cosine potential, has zero classical energy. The soliton excitation connects two adjacent classical ground states, and according to the above estimate the gap of soliton \( E_G \) is of the order \( \sqrt{\epsilon^2 \pi \epsilon a} = \sqrt{E_{\text{Coul}}} \). Since we expect \( E_{\text{Coul}} > t \) the Coulomb interaction enhances the value of band gap significantly.

In summary, we have studied excitons in 1D narrow-gap semiconductors of two anticrossing quantum Hall edges. According to our perturbative approach the exciton state may lie in the gap when \( \delta_{\text{ex}} < 1 \). Our study indicates that many-body interactions enhances the value of band gap significantly. Our result suggests that an exciton instability may occur when \( \delta_{\text{ex}} > 1 \). However, it is desirable to calculate how the higher-order corrections neglected in our perturbative calculation may change this condition. The actual values of \( \nu \) and \( t \) in anticrossing edges states are not known well and it is difficult to estimate precisely the actual value of \( \delta_{\text{ex}} \). Experimental observation of excitons and the investigation of exciton instability would be most interesting.

We are grateful to A. B. Zamolodchikov and M.P.A. Fisher for useful comments. H.C.L. was supported by the Korea Science and Engineering Foundation (KOSEF) through Grant No. 1999-2-11400-005-5, and by the Ministry of Education through Brain Korea 21 SNU-SKKU Program. S.R.E.Y. was supported by ‘‘Grant No. (1999-2-112-001-5) from the KOSEF and the KOSEF Quantum-functional Semiconductor Research Center at Dongguk University.’’

6 There are several small lengths in this problem: the transverse dimension of the quasi-1D system \( w \), the magnetic length \( l \), and the width of the barrier \( a \). They are all of order 100 Å. In this paper they will all be denoted by \( a \).
12 In our estimate it is likely that \( 0.1 < \delta_{\text{ex}} < 1 \).